

## SRINIVASA RAMANUJAN, 1887—1920.

## I.

SRINIVASA RAMANUJAN, who died at Kumbakonam on April 26, 1920, was elected a Fellow of the Society in 1918. He was not a man who talked much about himself, and until recently I knew very little of his early life. Two notices, by P. V. Seshu Aiyar and R. Ramachandra Rao, two of the most devoted of Ramanujan's Indian friends, have been published recently in the 'Journal of the Indian Mathematical Society'; and Sir Francis Spring has very kindly placed at my disposal an article which appeared in the 'Madras Times' of April 5, 1919. From these sources of information I can now supply a good many details with which I was previously unacquainted. Ramanujan (Srinivasa Iyengar Ramanuja Iyengar, to give him for once his proper name) was born on December 22, 1887, at Erode in southern India. His father was an accountant (*gumasta*) to a cloth merchant at Kumbakonam, while his maternal grandfather had served as *amin* in the Munsiff's (or local judge's) Court at Erode. He first went to school at five, and was transferred before he was seven to the Town High School at Kumbakonam, where he held a "free scholarship," and where his extraordinary powers appear to have been recognised immediately. "He used", so writes an old schoolfellow to Mr. Seshu Aiyar, "to borrow Carr's 'Synopsis of Pure Mathematics' from the College library, and delight in verifying some of the formulæ given there. . . . He used to entertain his friends with his theorems and formulæ, even in those early days. . . . He had an extraordinary memory, and could easily repeat the complete lists of Sanscrit roots (*atmanepada* and *parasmepada*); he could give the values of  $\sqrt{2}$ ,  $\pi$ ,  $e$ , . . . to any number of decimal places. . . . In manners, he was simplicity itself. . . ."

He passed his matriculation examination to the Government College at Kumbakonam in 1904, and secured the "Junior Subraniam Scholarship". Owing to weakness in English, he failed in his next examination and lost his scholarship, and left Kumbakonam, first for Vizagapatam and then for Madras. Here he presented himself for the "First Examination in Arts" in December, 1906, but failed and never tried again. For the next few years he continued his independent work in mathematics, "jotting down his results in two good-sized note-books"; I have one of these note-books in my possession still. In 1909 he married, and it became necessary for him to find some permanent employment. I quote Mr. Seshu Aiyar:—

To this end, he went to Tirukoilur, a small sub-division town in South Arcot District, to see Mr. V. Ramaswami Aiyar, the founder of the Indian Mathematical Society, but Mr. Aiyar, seeing his wonderful gifts, persuaded him to go to Madras. It was then after some four years' interval that Mr. Ramanujan met me at Madras, with his two

well-sized note-books referred to above. I sent Ramanujan with a note of recommendation to that true lover of mathematics, Dewan Bahadur R. Ramachandra Rao who was then District Collector at Nellore, a small town some eighty miles north of Madras. Mr. Rao sent him back to me saying it was cruel to make an intellectual giant like Ramanujan rot at a mofussil station like Nellore, and recommended his stay at Madras, generously undertaking to pay Mr. Ramanujan's expenses for a time. This was in December, 1910. After a while, other attempts to obtain for him a scholarship having failed, and Ramanujan himself being unwilling to be a burden on anybody for any length of time, he decided to take up a small appointment under the Madras Port Trust in 1911.

But he never slackened his work at mathematics. His earliest contribution to the 'Journal of the Indian Mathematical Society' was in the form of questions communicated by me in vol. 3 (1911). His first long article on "Some Properties of Bernoulli's Numbers" was published in the December number of the same volume. Mr. Ramanujan's methods were so terse and novel, and his presentation was so lacking in clearness and precision, that the ordinary reader, unaccustomed to such intellectual gymnastics, could hardly follow him. This particular article was returned more than once by the editor before it took a form suitable for publication. It was during this period that he came to me one day with some theorems on Prime Numbers, and when I referred him to Hardy's Tract on 'Orders of Infinity,' he observed that Hardy had said on p. 36 of his Tract "the exact order of  $\rho(x)$  [defined by the equation

$$\rho(x) = \pi(x) - \int_2^x \frac{dt}{\log t},$$

where  $\pi(x)$  denotes the number of primes less than  $x$ ] has not yet been determined," and that he himself had discovered a result which gave the order of  $\rho(x)$ . On this I suggested that he might communicate his result to Mr. Hardy, together with some more of his results.

This passage brings me to the beginning of my own acquaintance with Ramanujan. But before I say anything about the letters which I received from him, and which resulted ultimately in his journey to England, I must add a little more about his Indian career. Dr. G. T. Walker, F.R.S., Head of the Meteorological Department, and formerly Fellow and Mathematical Lecturer of Trinity College, Cambridge, visited Madras for some official purpose some time in 1912, and Sir Francis Spring, K.C.I.E., the Chairman of the Madras Port Authority, called his attention to Ramanujan's work. Dr. Walker was far too good a mathematician not to recognise its quality, little as it had in common with his own. He brought Ramanujan's case to the notice of the Government and the University of Madras. A research studentship, "Rs. 75 *per mensem* for a period of two years", was awarded him, and he became, and remained for the rest of his life, a professional mathematician.

## II.

Ramanujan wrote to me first on January 16, 1913, and at fairly regular intervals until he sailed for England in 1914. I do not believe that his letters were entirely his own. His knowledge of English, at that stage of his life, could scarcely have been sufficient, and there is an occasional phrase which is hardly characteristic. Indeed, I seem to remember his telling me

that his friends had given him some assistance. However, it was the mathematics that mattered, and that was very emphatically his.

“ Dear Sir,

Madras, January 16, 1913.

“ I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about twenty-three years of age. I have have had no university education, but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at mathematics. I have not trodden through the conventional regular course which is followed in a university course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general, and the results I get are termed by the local mathematicians as ‘startling’.

“ Just as in elementary mathematics you give a meaning to  $a^n$  when  $n$  is negative and fractional to conform to the law which holds when  $n$  is a positive integer, similarly the whole of my investigations proceed on giving a meaning to Eulerian Second Integral for all values of  $n$ . My friends who have gone through the regular course of university education tell me that  $\int_0^{\infty} x^{n-1} e^{-x} dx = \Gamma(n)$  is true only when  $n$  is positive. They say that this integral relation is not true when  $n$  is negative. Supposing this is true only for positive values of  $n$ , and also supposing the definition  $n\Gamma(n) = \Gamma(n+1)$  to be universally true, I have given meanings to these integrals, and under the conditions I state the integral is true for all values of  $n$  negative and fractional. My whole investigations are based upon this, and I have been developing this to a remarkable extent, so much so that the local mathematicians are not able to understand me in my higher flights.

“ Very recently I came across a tract published by you styled ‘Orders of Infinity’, in p. 36 of which I find a statement that no definite expression has been as yet found for the no of prime nos less than any given number. I have found an expression which very nearly approximates to the real result, the error being negligible. I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value, I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get, but I have indicated to the lines on which I proceed. Being inexperienced, I would very highly value any advice you give me. Requesting to be excused for the trouble I give you.

“ I remain, Dear sir, Yours truly,

“ S. RAMANUJAN.

“ P.S.—My address is S. Ramanujan, Clerk Accounts Department, Port Trust, Madras, India.”

I quote now from the “papers enclosed,” and from later letters.

“The following are a few examples from my theorems :—

(1) The nos of the form  $2^p 3^q$  less than  $n = \frac{1}{2} \frac{\log(2n) \log(3n)}{\log 2 \log 3}$  where  $p$  and  $q$  may have any positive integral value including 0.

(2) Let us take all nos containing an odd no of dissimilar prime divisors, viz. :—

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 30, 31, 37, 41, 42, 43, 47, etc.

(a) The no of such nos less than  $n = \frac{3n}{\pi^2}$ .

(b)  $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots + \frac{1}{30^2} + \frac{1}{31^2} + \dots = \frac{9}{2\pi^2}$ .

(c)  $\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots = \frac{15}{2\pi^4}$ .

(3) Let us take the no of divisors of natural nos, viz. :—

1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, etc. (1 having 1 divisor, 2 having 2, 3 having 2, 4 having 3, 5 having 2, etc.).

The sum of such nos to  $n$  terms

$$= n(2\gamma - 1 + \log n) + \frac{1}{2} \text{ of the no of divisors of } n,$$

where  $\gamma = 0.5772156649 \dots$ , the Eulerian Constant.

(4) 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, etc., are nos which are either themselves sqq. or which can be expressed as the sum of two sqq.

“The no of such nos greater than  $A$  and less than  $B$

$$= K \int_A^B \frac{dx}{\sqrt{\log x}} + \theta(B) \quad \text{where } K = 0.764 \dots$$

and  $\theta(x)$  is very small when compared with the previous integral.  $K$  and  $\theta(x)$  have been exactly found though complicated. . . .”

It may be well that I should interpolate here a few remarks concerning Ramanujan's researches in this particular field.

Ramanujan's theory of primes was vitiated by his ignorance of the theory of functions of a complex variable. It was (so to say) what the theory might be if the Zeta-function had no complex zeros. His methods of proof depended upon a wholesale use of divergent series. He disregarded entirely all the difficulties which are involved in the interchange of double limit operations; he did not distinguish, for example, between the sum of a series  $\sum a_n$  and the value of the Abelian limit

$$\lim_{x \rightarrow 1} \sum a_n x^n,$$

or that of any other limit which might be used for similar purposes by a

modern analyst. There are regions of mathematics in which the precepts of modern rigour may be disregarded with comparative safety, but the Analytic Theory of Numbers is not one of them, and Ramanujan's Indian work on primes, and on all the allied problems of the theory, was definitely wrong. That his proofs should have been invalid was only to be expected. But the mistakes went deeper than that, and many of the actual results were false. He had obtained the dominant terms of the classical formulæ, although by invalid methods; but none of them are such close approximations as he supposed.

This may be said to have been Ramanujan's one great failure. And yet I am not sure that, in some ways, his failure was not more wonderful than any of his triumphs. Consider, for example, problem (4). The dominant term is

$$\frac{KB}{\sqrt{\log B}} \quad (\alpha)$$

(I adhere to Ramanujan's notation): this result was first obtained by Landau in 1908. The error is of order  $B(\log B)^{-\frac{1}{2}}$ ; and this is so far in agreement with Ramanujan's assertion. Ramanujan, however, implies much more,\* and what he implies is definitely false: his integral does not represent the number of numbers in question more accurately than Landau's formula ( $\alpha$ ). However, Ramanujan had none of Landau's weapons at his command; he had never seen a French or German book; his knowledge even of English was insufficient to enable him to qualify for a degree. It is sufficiently marvellous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve, and of which the solution is incomplete to the present day.

“ . . . IV. Theorems on integrals. The following are a few examples:—

$$(1) \int_0^\infty \frac{1 + \left(\frac{x}{b+1}\right)^2}{1 + \left(\frac{x}{a}\right)^2} \cdot \frac{1 + \left(\frac{x}{b+2}\right)^2}{1 + \left(\frac{x}{a+1}\right)^2} \dots \text{etc. } dx$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)} \cdot \frac{\Gamma(b+1)}{\Gamma(b+\frac{1}{2})} \cdot \frac{\Gamma(b-a+\frac{1}{2})}{\Gamma(b-a+1)}.$$

.....

$$(3) \text{ If } \int_0^\infty \frac{\cos nx}{e^{2\pi\sqrt{x}} - 1} dx = \phi(n),$$

$$\text{then } \int_0^\infty \frac{\sin nx}{e^{2\pi\sqrt{x}} - 1} dx = \phi(n) - \frac{1}{2n} + \phi\left(\frac{\pi^2}{n}\right) \sqrt{\frac{2\pi^3}{n^3}}.$$

\* See his statement concerning the order of  $\theta(x)$ , quoted below from his letter of February 27, 1913.

$\phi(n)$  is a complicated function. The following are certain special values :

$$\phi(0) = \frac{1}{12}; \quad \phi\left(\frac{\pi}{2}\right) = \frac{1}{4\pi}; \quad \phi(\pi) = \frac{2-\sqrt{2}}{8}; \quad \phi(2\pi) = \frac{1}{16};$$

$$\phi\left(\frac{2\pi}{5}\right) = \frac{8-8\sqrt{5}}{16}; \quad \phi\left(\frac{\pi}{5}\right) = \frac{6+\sqrt{5}}{4} - \frac{5\sqrt{10}}{8}; \quad \phi(\infty) = 0;$$

$$\phi\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \sqrt{3}\left(\frac{3}{16} - \frac{1}{8\pi}\right).$$

$$(4) \int_0^{\infty} \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\dots \text{etc.}} = \frac{\pi}{2(1+r+r^3+r^5+r^7+\dots \text{etc.})},$$

where 1, 3, 6, 10, etc., are sums of natural nos.

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$$(6) \int_0^{\infty} \frac{\tan^{-1} \frac{2nz}{n^2+x^2-z^2}}{e^{2\pi z}-1} dz \text{ can be exactly found if } 2n \text{ is any integer and}$$

$x$  any quantity."

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"V. Theorems on summation of series,\* e.g.,

$$(1) \frac{1}{1^3} \cdot \frac{1}{2} + \frac{1}{2^3} \cdot \frac{1}{2^2} + \frac{1}{3^3} \cdot \frac{1}{2^3} + \frac{1}{4^3} \cdot \frac{1}{2^4} + \text{etc.}$$

$$= \frac{1}{6}(\log 2)^3 - \frac{\pi^2}{12} \log 2 + \left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \text{etc.}\right).$$

$$(2) 1 + 9 \cdot \left(\frac{1}{4}\right)^4 + 17 \cdot \left(\frac{1 \cdot 5}{4 \cdot 8}\right)^4 + 25 \cdot \left(\frac{1 \cdot 5 \cdot 9}{4 \cdot 8 \cdot 12}\right)^4 + \text{etc.} = \frac{2\sqrt{2}}{\sqrt{\pi} \cdot \{\Gamma(\frac{3}{4})\}^2}.$$

$$(3) 1 - 5 \cdot \left(\frac{1}{2}\right)^3 + 9 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^3 - \text{etc.} = \frac{2}{\pi}.$$

$$(4) \frac{1^{13}}{e^{2\pi}-1} + \frac{2^{13}}{e^{4\pi}-1} + \frac{3^{13}}{e^{6\pi}-1} + \text{etc.} = \frac{1}{24}.$$

$$(5) \frac{\coth \pi}{1^7} + \frac{\coth 2\pi}{2^7} + \frac{\coth 3\pi}{3^7} + \text{etc.} = \frac{19\pi^7}{56700}.$$

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$$(11) \frac{2}{3} \int_0^1 \frac{\tan^{-1} x}{x} dx - \int_0^{2-\sqrt{3}} \frac{\tan^{-1} x}{x} dx = \frac{\pi}{12} \log(2+\sqrt{3}).$$

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\* There is always more in one of Ramanujan's formulæ than meets the eye, as anyone who sets to work to verify those which look the easiest will soon discover. In some the interest lies very deep, in others comparatively near the surface; but there is not one which is not curious and entertaining.

“ VI. Theorems on transformation of series and integrals, e.g.,

$$(1) \pi \left( \frac{1}{2} - \frac{1}{\sqrt{1+\sqrt{3}}} + \frac{1}{\sqrt{3+\sqrt{5}}} - \frac{1}{\sqrt{5+\sqrt{7}}} + \text{etc.} \right) \\ = \frac{1}{1\sqrt{1}} - \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} - \text{etc.}$$

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(4) If  $\int_0^a \phi(p, x) \cos nx \, dx = \psi(p, n)$ , then

$$\frac{\pi}{2} \int_0^{\infty} \phi(p, x) \phi(q, nx) \, dx = \int_0^{\infty} \phi(q, x) \phi(p, nx) \, dx.$$

(5) If  $\alpha\beta = \pi$ , then  $\sqrt{\alpha} \int_0^{\infty} \frac{e^{-x^2}}{\cosh \alpha x} \, dx = \sqrt{\beta} \int_0^{\infty} \frac{e^{-x^2}}{\cosh \beta x} \, dx.$

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“ VII. Theorems on approximate integration and summation of series.

$$(1) 1^2 \log 1 + 2^2 \log 2 + 3^2 \log 3 + \dots + x^2 \log x \\ = \frac{x(x+1)(2x+1)}{6} \log x - \frac{x^3}{9} + \frac{1}{4\pi^2} \left( \frac{1}{1^3} + \frac{1}{3^3} + \dots \right) + \frac{x}{12} - \frac{1}{360x} + \dots$$

$$(2) 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^x}{x} \theta = \frac{e^x}{2}$$

where  $\theta = \frac{1}{3} + \frac{4}{135(x+k)}$  where  $k$  lies between  $\frac{8}{45}$  and  $\frac{2}{21}$ .

$$(3) 1 + \left( \frac{x}{1} \right)^5 + \left( \frac{x^2}{2} \right)^5 + \left( \frac{x^3}{3} \right)^5 + \text{etc.} = \frac{\sqrt{5}}{4\pi^2} \cdot \frac{e^{5x}}{5x^2 - x + \theta},$$

where  $\theta$  vanishes when  $x = \infty$ .

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$$(5) \frac{1}{1001} + \frac{1}{1002^2} + \frac{3}{1003^3} + \frac{4^2}{1004^4} + \frac{5^3}{1005^5} + \text{etc.} \\ = \frac{1}{1000} - 10^{-440} \times 1.0125 \text{ nearly.}$$

$$(6) \int_0^a e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} - \frac{e^{-a^2}}{2a} + \frac{1}{a} + \frac{2}{2a} + \frac{3}{a} + \frac{4}{2a} + \text{etc.}$$

$$(7) \text{The coefficient of } x^n \text{ in } \frac{1}{1-2x+2x^4-2x^9+2x^{16}-\text{etc.}} \\ = \text{the nearest integer to } \frac{1}{4n} \left\{ \cosh(\pi\sqrt{n}) - \frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}} \right\}.*$$

\* This is quite untrue. But the formula is extremely interesting for a variety of reasons.

"IX. Theorems on continued fractions, a few examples are:—

$$(1) \frac{4}{x+2x} + \frac{1^2}{2x+2x} + \frac{3^2}{2x+2x} + \frac{5^2}{2x+2x} + \frac{7^2}{2x+2x} + \text{etc.} = \left\{ \frac{\Gamma\left(\frac{x+1}{4}\right)}{\Gamma\left(\frac{x+3}{4}\right)} \right\}^2$$

.....

$$(4) \text{ If } u = \frac{x}{1+1} + \frac{x^5}{1+1} + \frac{x^{10}}{1+1} + \frac{x^{15}}{1+1} + \frac{x^{20}}{1+1} + \text{etc.}$$

and

$$v = \frac{\sqrt[5]{x}}{1+1} + \frac{x}{1+1} + \frac{x^2}{1+1} + \frac{x^3}{1+1} + \text{etc.}$$

then

$$v^5 = u \cdot \frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}$$

$$(5) \frac{1}{1+1} + \frac{e^{-2\pi}}{1+1} + \frac{e^{-4\pi}}{1+1} + \frac{e^{-6\pi}}{1+1} + \text{etc.} = \left( \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5+1}}{2} \right)^5 \sqrt[5]{e^{2\pi}}$$

$$(6) \frac{1}{1-1} + \frac{e^{-\pi}}{1+1} + \frac{e^{-2\pi}}{1+1} + \frac{e^{-3\pi}}{1+1} + \text{etc.} = \left( \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5-1}}{2} \right)^5 \sqrt[5]{e^{\pi}}$$

$$(7) \frac{1}{1+1} + \frac{e^{-\pi\sqrt{n}}}{1+1} + \frac{e^{-2\pi\sqrt{n}}}{1+1} + \frac{e^{-3\pi\sqrt{n}}}{1+1} + \text{etc.} \text{ can be exactly found if } n \text{ be any}$$

positive rational quantity. . . ."

February 27, 1913.

" . . . I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed. . . . I find in many a place in your letter rigorous proofs are required, and you ask me to communicate the methods of proof. . . . I told him\* that the sum of an infinite no of terms of the series  $1+2+3+4+\dots = -\frac{1}{2}$  under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. . . . What I tell you is this. Verify the results I give, and, if they agree with your results . . . you should at least grant that there may be some truths in my fundamental basis. . . .

"To preserve my brains I want food, and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship, either from the University or from Government. . . .

$$1. \text{ The no of prime nos less than } e^a = \int_0^a \frac{a^x dx}{x S_{x+1} \Gamma(x+1)}$$

where

$$S_{x+1} = \frac{1}{1^{x+1}} + \frac{1}{2^{x+1}} + \dots$$

\* Referring to a previous correspondence.

2. The no of prime nos less than  $n =$

$$\frac{2}{\pi} \left\{ \frac{2}{B_2} \left( \frac{\log n}{2\pi} \right) + \frac{4}{3B_4} \left( \frac{\log n}{2\pi} \right)^3 + \frac{6}{5B_6} \left( \frac{\log n}{2\pi} \right)^5 + \text{etc.} \right.$$

where  $B_2 = \frac{1}{6}$ ;  $B_4 = \frac{1}{30}$ , etc., the Bernoullian nos. . . .

The order of  $\theta(x)$  which you asked in your letter is  $\sqrt{\left(\frac{x}{\log x}\right)}$ .

(1) If  $F(x) = \frac{1}{1} + \frac{x}{1} + \frac{x^2}{1} + \frac{x^3}{1} + \frac{x^4}{1} + \frac{x^5}{1} + \text{etc.}$

then  $\left\{ \frac{\sqrt{5+1}}{2} + e^{-2\alpha/5} F(e^{-2\alpha}) \right\} \left\{ \frac{\sqrt{5+1}}{2} + e^{-2\beta/5} F(e^{-2\beta}) \right\} = \frac{5+\sqrt{5}}{2}$ ,

with the condition  $\alpha\beta = \pi^2$ . . . .

e.g.  $\frac{1}{1} + \frac{e^{-2\pi\sqrt{5}}}{1} + \frac{e^{-4\pi\sqrt{5}}}{1} + \text{etc.} \dots = e^{\pi/\sqrt{5}} \left\{ \frac{\sqrt{5}}{1 + \sqrt[5]{5^{\frac{5}{2}} \left( \frac{\sqrt{5}-1}{2} \right)^5} - 1} - \frac{\sqrt{5+1}}{2} \right\}$

The above theorem is a particular case of a theorem on the c.f.

$$\frac{1}{1} + \frac{ax}{1} + \frac{ax^2}{1} + \frac{ax^3}{1} + \frac{ax^4}{1} + \frac{ax^5}{1} + \text{etc.}$$

which is a particular case of the c.f.

$$\frac{1}{1} + \frac{ax}{1+bx} + \frac{ax^2}{1+bx^2} + \frac{ax^3}{1+bx^3} + \text{etc.}$$

which is a particular case of a general theorem on c.f.

(2) i.  $4 \int_0^\infty \frac{x e^{-x\sqrt{5}}}{\cosh x} dx = \frac{1}{1} + \frac{1^2}{1} + \frac{1^2}{1} + \frac{2^2}{1} + \frac{2^2}{1} + \frac{3^2}{1} + \frac{3^2}{1} + \text{etc.}$

ii.  $4 \int_0^\infty \frac{x^2 e^{-x\sqrt{3}}}{\cosh x} dx = \frac{1}{1} + \frac{1^3}{1} + \frac{1^3}{3} + \frac{2^3}{1} + \frac{2^3}{5} + \frac{3^3}{1} + \frac{3^3}{7} + \text{etc.}$

(5)  $\frac{1^5}{e^{2\pi}-1} \frac{1}{2500+1^4} + \frac{2^5}{e^{4\pi}-1} \frac{1}{2500+2^4} + \dots = \frac{123826979}{6306456} - \frac{25\pi}{4} \coth^2 5\pi$ .

(6) If  $v = \frac{x}{1} + \frac{x^3+x^6}{1} + \frac{x^6+x^{12}}{1} + \frac{x^9+x^{18}}{1} + \text{etc.}$

then i.  $x \left( 1 + \frac{1}{v} \right) = \frac{1+x+x^3+x^6+x^{10} + \text{etc.}}{1+x^9+x^{27}+x^{54}+x^{90} + \text{etc.}}$

ii.  $x^3 \left( 1 + \frac{1}{v^3} \right) = \left( \frac{1+x+x^3+x^6+^{10} + \text{etc.}}{1+x^3+x^9+x^{18}+x^{30} + \text{etc.}} \right)^4$ .

$$(8) \int_0^{\infty} \frac{1+ab^2x^2}{1+x^2} \frac{1+ab^4x^2}{1+b^2x^2} \dots dx = \frac{\pi}{2(1+b+b^3+b^5+\dots)} \frac{1-ab^2}{1-ab} \frac{1-ab^4}{1-ab^3} \dots$$

.....  
 (12) If  $\frac{1}{2}\pi\alpha = \log \tan \left(\frac{1}{4}\pi + \frac{1}{4}\pi\beta\right)$ , then

$$\frac{1^2 + \alpha^2}{1^2 - \beta^2} \left(\frac{3^2 - \beta^2}{3^2 + \alpha^2}\right)^3 \left(\frac{5^2 + \alpha^2}{5^2 - \beta^2}\right)^5 \left(\frac{7^2 - \beta^2}{7^2 + \alpha^2}\right)^7 \dots = e^{\frac{1}{2}\pi\alpha\beta}.$$

.....  
 (16) If  $F(\alpha, \beta) = \tan^{-1} \left( \frac{\alpha}{x} + \frac{\beta^2 + k^2}{x} + \frac{\alpha^2 + (2k)^2}{x} + \frac{\beta^2 + (3k)^2}{x} \dots \right)$ , then

$$F(\alpha, \beta) + F(\beta, \alpha) = 2F\left(\frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2}\right).$$

(17) If  $F(k) = 1 + \left(\frac{1}{2}\right)^2 k + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^2 + \dots$  and  $F(1-k) = \sqrt{(210) F(k)}$ ,

$$\text{then } k = (\sqrt{2}-1)^4 (2-\sqrt{3})^2 (\sqrt{7}-\sqrt{6})^4 (8-3\sqrt{7})^2 (\sqrt{10}-3)^4 (4-\sqrt{15})^4 \\ \times (\sqrt{15}-\sqrt{14})^2 (6-\sqrt{35})^2.$$

(18) If  $F(\alpha) = 1 + \frac{1 \cdot 2}{3^2} \alpha + \frac{1 \cdot 2 \cdot 4 \cdot 5}{3^2 \cdot 6^2} \alpha^2 + \dots$

and 
$$\frac{F(1-\alpha)}{F(\alpha)} = 5 \frac{F(1-\beta)}{F(\beta)},$$

then  $\sqrt[3]{\alpha\beta} + \sqrt[3]{\{(1-\alpha)(1-\beta)\}} + 3\sqrt[5]{\alpha\beta(1-\alpha)(1-\beta)} = 1.$

.....

April 17, 1913.

"... I am a little pained to see what you have written. . . ." \* I am not in the least apprehensive of my method being utilised by others. On the contrary, my method has been in my possession for the last eight years, and I have not found anyone to appreciate the method. As I wrote in my last letter, I have found a sympathetic friend in you, and I am willing to place unreservedly in your hands what little I have. It was on account of the novelty of the method I have used that I am a little diffident even now to communicate my own way of arriving at the expressions I have already given. . . .

"... I am glad to inform you that the local University has been pleased to grant me a scholarship of £60 per annum for two years, and this was at the instance of Dr. Walker, F.R.S., Head of the Meteorological Department in India, to whom my thanks are due. . . . I request you to convey my thanks also to Mr. Littlewood, Dr. Barnes, Mr. Berry, and others who take an interest in me. . . ."

\* Ramanujan might very reasonably have been reluctant to give away his secrets to an English mathematician, and I had tried to reassure him on this point as well as I could.

## III.

It is unnecessary to repeat the story of how Ramanujan was brought to England. There were serious difficulties; and the credit for overcoming them is due primarily to Prof. E. H. Neville, in whose company Ramanujan arrived in April, 1914. He had a scholarship from Madras of £250, of which £50 was allotted to the support of his family in India, and an exhibition of £60 from Trinity. For a man of his almost ludicrously simple tastes, this was an ample income; and he was able to save a good deal of money, which was badly wanted later. He had no duties, and could do as he pleased; he wished, indeed, to qualify for a Cambridge degree as a research student, but this was a formality. He was now, for the first time in his life, in a really comfortable position, and could devote himself to his researches without anxiety.

There was one great puzzle. What was to be done in the way of teaching him modern mathematics? The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations, and theorems of complex multiplication, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world, who had found for himself the functional equation of the Zeta-function, and the dominant terms of many of the most famous problems in the analytic theory of numbers; and he had never heard of a doubly periodic function or of Cauchy's theorem, and had, indeed, but the vaguest idea of what a function of a complex variable was. His ideas as to what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account.

It was impossible to ask such a man to submit to systematic instruction, to try to learn mathematics from the beginning once more. I was afraid, too, that, if I insisted unduly on matters which Ramanujan found irksome, I might destroy his confidence or break the spell of his inspiration. On the other hand, there were things of which it was impossible that he should remain in ignorance. Some of his results were wrong, and in particular those which concerned the distribution of primes, to which he attached the greatest importance. It was impossible to allow him to go through life supposing that all the zeros of the Zeta-function were real. So I had to try to teach him, and in a measure I succeeded, though obviously I learnt from him much more than he learnt from me. In a few years' time he had a very tolerable knowledge of the theory of functions and the analytic theory of numbers. He was never a mathematician of the modern school, and it was hardly desirable that he should become one; but he knew when he had proved a theorem and when he had not. And his flow of original ideas showed no symptom of abatement.

I should add here a word about Ramanujan's interests outside mathematics. Like his mathematics, they showed the strangest contrasts. He had very little interest, I should say, in literature as such, or in art, though he could tell good literature from bad. On the other hand, he was a keen philosopher, of what appeared, to followers of the modern Cambridge school, a rather nebulous kind, and an ardent politician, of a pacifist and ultra-radical type. He adhered, with a severity most unusual in Indians resident in England, to the religious observances of his caste; but his religion was a matter of observance and not of intellectual conviction, and I remember well his telling me (much to my surprise) that all religions seemed to him more or less equally true. Alike in literature, philosophy, and mathematics, he had a passion for what was unexpected, strange, and odd; he had quite a small library of books by circle-squarers and other cranks.

It was in the spring of 1917 that Ramanujan first appeared to be unwell. He went into the Nursing Home at Cambridge in the early summer, and was never out of bed for any length of time again. He was in sanatoria at Wells, at Matlock, and in London, and it was not until the autumn of 1918 that he showed any decided symptom of improvement. He had then resumed active work, stimulated perhaps by his election to the Royal Society, and some of his most beautiful theorems were discovered about this time. His election to a Trinity Fellowship was a further encouragement. Early in 1919 he had recovered, it seemed, sufficiently for the voyage home to India, and the best medical opinion held out hopes of a permanent restoration. I was rather alarmed by not hearing from him for a considerable time; but a letter reached me in February, 1920, from which it appeared that he was still active in research.

University of Madras,  
January 12, 1920.

"I am extremely sorry for not writing you a single letter up to now. . . . I discovered very interesting functions recently which I call 'Mock'  $\mathcal{J}$ -functions. Unlike the 'False'  $\mathcal{J}$ -functions (studied partially by Prof. Rogers in his interesting paper), they enter into mathematics as beautifully as the ordinary  $\mathcal{J}$ -functions. I am sending you with this letter some examples. . . .

*Mock  $\mathcal{J}$ -functions*

$$\phi(q) = 1 + \frac{q}{1+q^2} + \frac{q^4}{(1+q^2)(1+q^4)} + \dots$$

$$\psi(q) = \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^3)} + \frac{q^9}{(1-q)(1-q^3)(1-q^5)} + \dots$$

*Mock  $\mathcal{J}$ -functions (of 5th order)*

$$f(q) = 1 + \frac{q}{1+q} + \frac{q^4}{(1+q)(1+q^2)} + \frac{q^9}{(1+q)(1+q)(1+q^3)} + \dots$$

Mock  $\mathcal{D}$ -functions (of 7th order)

$$(i) \quad 1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)} + \dots$$

.....”

He said little about his health, and what he said was not particularly discouraging; and I was quite unprepared for the news of his death.

#### IV.

Ramanujan published the following papers in Europe:—

- (1) “Some Definite Integrals,” ‘Messenger of Mathematics,’ vol. 44, pp. 10–18 (1914).
- (2) “Some Definite Integrals connected with Gauss’s sums,” *ibid.*, pp. 75–85.
- (3) “Modular Equations and Approximations to  $\pi$ ,” ‘Quarterly Journal of Mathematics,’ vol. 45, pp. 350–372 (1914).
- (4) “New Expressions for Riemann’s Functions  $\zeta(s)$  and  $\Xi(t)$ ,” *ibid.*, vol. 46, pp. 253–261 (1915).
- (5) “On Certain Infinite Series,” ‘Messenger of Mathematics,’ vol. 45, pp. 11–15 (1915).
- (6) “Summation of a Certain Series,” *ibid.*, pp. 157–160.
- (7) “Highly Composite Numbers,” ‘Proc. London Math. Soc.,’ ser. 2, vol. 14, pp. 347–409 (1915).
- (8) “Some Formulæ in the Analytic Theory of Numbers,” ‘Messenger of Mathematics,’ vol. 45, pp. 81–84 (1916).
- (9) “On Certain Arithmetical Functions,” ‘Trans. Cambridge Phil. Soc.,’ vol. 22, No. 9, pp. 159–184 (1916).
- (10) “Some Series for Euler’s Constant,” ‘Messenger of Mathematics,’ vol. 46, pp. 73–80 (1916).
- (11) “On the Expression of Numbers in the Form  $ax^2 + by^2 + cz^2 + dt^2$ ,” ‘Proc. Cambridge Phil. Soc.,’ vol. 19, pp. 11–21 (1917).
- \*(12) “Une formule asymptotique pour le nombre des partitions de  $n$ ,” ‘Comptes Rendus,’ January 2, 1917.
- \*(13) “Asymptotic Formulæ concerning the Distribution of Integers of various Types,” ‘Proc. London Math. Soc.,’ ser. 2, vol. 16, pp. 112–132 (1917).
- \*(14) “The Normal Number of Prime Factors of a Number  $n$ ,” ‘Quarterly Journal of Mathematics,’ vol. 48, pp. 76–92 (1917).
- \*(15) “Asymptotic Formulæ in Combinatory Analysis,” ‘Proc. London Math. Soc.,’ ser. 2, vol. 17, pp. 75–115 (1918).
- \*(16) “On the Coefficients in the Expansions of Certain Modular Functions,” ‘Roy. Soc. Proc.,’ A, vol. 95, pp. 144–155 (1918).
- (17) “On Certain Trigonometrical Sums and their Applications in the Theory of Numbers,” ‘Trans. Cambridge Phil. Soc.,’ vol. 22, pp. 259–276 (1918).
- (18) “Some Properties of  $p(n)$ , the Number of Partitions of  $n$ ,” ‘Proc. Cambridge Phil. Soc.,’ vol. 19, pp. 207–210 (1919).
- (19) “Proof of Certain Identities in Combinatory Analysis,” *ibid.*, pp. 214–216.
- (20) “A Class of Definite Integrals,” ‘Quarterly Journal of Mathematics,’ vol. 48, pp. 294–309 (1920).
- (21) “Congruence Properties of Partitions,” ‘Math. Zeitschrift,’ vol. 9, pp. 147–153 (1921).

Of these, those marked with an asterisk were written in collaboration with me, and (21) is a posthumous extract from a much larger unpublished

manuscript in my possession.† He also published a number of short notes in the 'Records of Proceedings at Meetings' of the London Mathematical Society, and in the 'Journal of the Indian Mathematical Society.' The complete list of these is as follows:—

*Records of Proceedings at Meetings.*

- \* (22) "Proof that almost all Numbers  $n$  are Composed of about  $\log \log n$  prime factors," December 14, 1916.  
 \*(23) "Asymptotic Formulæ in Combinatory Analysis," March 1, 1917.  
 (24) "Some Definite Integrals," Jan. 17, 1918.  
 (25) "Congruence Properties of Partitions," March 13, 1919.  
 (26) "Algebraic Relations between certain Infinite Products," March 13, 1919.

*Journal of the Indian Mathematical Society.*

(A) Articles and Notes.

- (27) "Some Properties of Bernoulli's Numbers," vol. 3, pp. 219–235 (1911).  
 (28) "On Q. 330 of Prof. Sanjaná," vol. 4, pp. 59–61 (1912).  
 (29) "A Set of Equations," vol. 4, pp. 94–96 (1912).  
 (30) "Irregular Numbers," vol. 5, pp. 105–107 (1913).  
 (31) "Squaring the Circle," vol. 5, pp. 132–133 (1913).  
 (32) "On the integral  $\int_0^x \arctan t \cdot \frac{dt}{t}$ ," vol. 7, pp. 93–96 (1915).  
 (33) "On the Divisors of a Number," vol. 7, pp. 131–134 (1915).  
 (34) "The Sum of the Square Roots of the First  $n$  Natural Numbers," vol. 7, pp. 173–175 (1915).  
 (35) "On the Product  $\pi \left[ 1 + \frac{x^2}{(a+nd)^2} \right]$ ," vol. 7, pp. 209–212 (1915).  
 (36) "Some Definite Integrals," vol. 11, pp. 81–88 (1919).  
 (37) "A Proof of Bertrand's Postulate," vol. 11, pp. 181–183 (1919).  
 (38) (Communicated by S. Narayana Aiyar), vol. 3, p. 60 (1911).

(B) Questions proposed and solved.

Nos. 260, 261, 283, 289, 294, 295, 298, 308, 353, 358, 386, 427, 441, 464, 489, 507, 541, 546, 571, 605, 606, 629, 642, 666, 682, 700, 723, 724, 739, 740, 753, 768, 769, 783, 785.

(C) Questions proposed but not solved as yet.

Nos. 284, 327, 359, 387, 441, 463, 469, 524, 525, 526, 584, 661, 662, 681, 699, 722, 738, 754, 770, 784, 1049, 1070, and 1076.

Finally, I may mention the following writings by other authors, concerned with Ramanujan's work:—

- "Proof of a Formula of Mr. Ramanujan," by G. H. Hardy, 'Messenger of Mathematics,' vol. 44, pp. 18–21 (1915).  
 "Mr. S. Ramanujan's Mathematical Work in England," by G. H. Hardy (Report to the University of Madras, 1916. Privately printed).  
 "On Mr. Ramanujan's Empirical Expansions of Modular Functions," by L. J. Mordell, 'Proc. Camb. Phil. Soc.,' vol. 19, pp. 117–124 (1917).  
 "On Mr. Ramanujan's Congruence Properties of  $p(n)$ ," by H. B. C. Darling, 'Proc. Cambridge Phil. Soc.,' vol. 19, pp. 217–218 (1919).

† All of Ramanujan's manuscripts passed through my hands, and I edited them very carefully for publication. The earlier ones I rewrote completely. I had no share of any kind in the results, except of course when I was actually a collaborator, or when explicit acknowledgment is made. Ramanujan was almost absurdly scrupulous in his desire to acknowledge the slightest help.

"Life Sketch of Ramanujan," editorial in the 'Journal of the Indian Math. Soc.,' vol. 11, p. 122 (1919).

"Note on the Parity of the Number which Enumerates the Partitions of a Number," by P. A. MacMahon, 'Proc. Cambridge Phil. Soc.,' vol. 20, pp. 281-283 (1921).

"Proof of certain Identities and Congruences Enunciated by S. Ramanujan," by H. B. C. Darling, 'Proc. London Math. Soc.,' ser. 2, vol. 19, pp. 350-372 (1921).

"On a Type of Modular Relation," by L. J. Rogers, *ibid.*, pp. 387-397.

It is plainly impossible for me, within the limits of a notice such as this, to attempt a reasoned estimate of Ramanujan's work. Some of it is very intimately connected with my own, and my verdict could not be impartial; there is much, too, that I am hardly competent to judge; and there is a mass of unpublished material, in part new and in part anticipated, in part proved and in part only conjectured, that still awaits analysis. But it may be useful if I state, shortly and dogmatically, what seems to me Ramanujan's finest, most independent, and most characteristic work.

His most remarkable papers appear to me to be (3), (7), (9), (17), (18), (19), and (21). The first of these is mainly Indian work, done before he came to England; and much of it had been anticipated. But there is much that is new, and, in particular, a very remarkable series of algebraic approximations to  $\pi$ . I may mention only the formulæ

$$\pi = \frac{63}{25} \frac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}}, \quad \frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2},$$

correct to 9 and 8 places of decimals respectively.

The long memoir (7) represents work, perhaps, in a backwater of mathematics, and is somewhat overloaded with detail; but the elementary analysis of "highly composite" numbers—numbers which have more divisors than any preceding number—is exceedingly remarkable, and shows very clearly Ramanujan's extraordinary mastery over the algebra of inequalities. Papers (9) and (17) should be read together, and in connection with Mr. Mordell's paper mentioned above; for Mr. Mordell afterwards proved a great deal that Ramanujan conjectured. They contain, in particular, very beautiful contributions to the theory of the representation of numbers by sums of squares. But I am inclined to think that it was in the theory of partitions, and the allied parts of the theories of elliptic functions and continued fractions, that Ramanujan shows at his very best. It is in papers (18), (19), and (21), and in the papers of Prof. Rogers and Mr. Darling, that I have quoted, that this side of his work (so far as it has been published) is to be found. It would be difficult to find more beautiful formulæ than the "Rogers-Ramanujan" identities, proved in (19); but here Ramanujan must take second place to Prof. Rogers; and, if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting a formula from (18), viz.,

$$p(4) + p(9)x + p(14)x^2 + \dots = 5 \frac{\{(1-x^5)(1-x^{10})(1-x^{15})\dots\}^5}{\{(1-x)(1-x^2)(1-x^3)\dots\}^6},$$

where  $p(n)$  is the number of partitions of  $n$ .

I have often been asked whether Ramanujan had any special secret; whether his methods differed in kind from those of other mathematicians; whether there was anything really abnormal in his mode of thought. I cannot answer these questions with any confidence or conviction; but I do not believe it. My belief is that all mathematicians think, at bottom, in the same kind of way, and that Ramanujan was no exception. He had, of course, an extraordinary memory. He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Mr. Littlewood (I believe) who remarked that "every positive integer was one of his personal friends." I remember once going to see him when he was lying ill at Putney. I had ridden in taxicab No. 1729, and remarked that the number (7.13.19) seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways." I asked him, naturally, whether he knew the answer to the corresponding problem for fourth powers; and he replied, after a moment's thought, that he could see no obvious example, and thought that the first such number must be very large. His memory, and his powers of calculation, were very unusual, but they could not reasonably be called "abnormal." If he had to multiply two large numbers, he multiplied them in the ordinary way; he would do it with unusual rapidity and accuracy, but not more rapidly or more accurately than any mathematician who is naturally quick and has the habit of computation. There is a table of partitions at the end of our paper (15). This was, for the most part, calculated independently by Ramanujan and Major MacMahon; and Major MacMahon was, in general, slightly the quicker and more accurate of the two.

It was his insight into algebraical formulæ, transformations of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He worked, far more than the majority of modern mathematicians, by induction from numerical examples; all of his congruence properties of partitions, for example, were discovered in this way. But with his memory, his patience, and his power of calculation, he combined also a power of generalisation, a feeling for form, and a capacity for rapid modification of his hypotheses, that was often really startling, and made him, in his own peculiar field, without a rival in his day.

It is often said that it is much more difficult now for a mathematician to be original than it was in the great days when the foundations of modern analysis were laid; and no doubt in a measure it is true. Opinions may differ as to the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on the mathematics of the future. It has not the simplicity and the inevitableness of the very greatest work; it would be greater if it were less strange. One gift it has which no one can deny, profound and invincible originality. He would probably have been a greater mathematician if he had been

caught and tamed a little in his youth; he would have discovered more that was new, and that, no doubt, of greater importance. On the other hand, he would have been less of a Ramanujan, and more of a European professor, and the loss might have been greater than the gain.

G. H. H.

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WOLDEMAR VOIGT, 1850—1919.

WOLDEMAR VOIGT was born at Leipzig in the year 1850. His school education was followed by attendance at the University there until his studies were interrupted by the outbreak of the Franco-German war of 1870, in which he served. His University course was afterwards completed at Königsberg in the years 1871-74. He here came under the influence of Franz Neumann, which determined largely the character of his own subsequent work. Alike in the type of subjects to which he devoted himself, in the formal elegance of his mathematical expositions, and in the severe precision of his style we can trace the inspiration of his illustrious teacher, for whom he retained a profound veneration. His abilities quickly gained recognition: he was made Extraordinary Professor at Königsberg in 1875, and was called to Göttingen as Ordinary Professor of Theoretical Physics in 1883. He held this post till his death on December 13, 1919.

The complexion of his scientific work is, on the whole, indicated by the title of his chair. It was, however, always in close relation to phenomena, and he carried out a classical series of measurements on the elasticity of crystals which demanded the utmost precision. For experimental work, except on a minute scale, he appears to have had, till a late period, scanty facilities. His favourite province, which he cultivated with a life-long enthusiasm, was the physical properties of crystals. In particular his determinations of elastic constants of various crystals may claim to have set at rest a historical controversy. The special form of molecular hypothesis on which the theories of Poisson and Cauchy were based involved the conclusion that certain relations must exist between the constants of a crystal whatever its classification. This would reduce the number of constants in a general scheme to fifteen. Voigt succeeded in proving that the relations in question were, in many cases, not even approximately fulfilled. This may be regarded as a vindication of the attitude of Green and his followers in this country who, avoiding special hypothesis, postulated only the principle of energy. Voigt himself did not accept this point of view as final, and indicated lines on which the old molecular hypothesis might be amended. A masterly