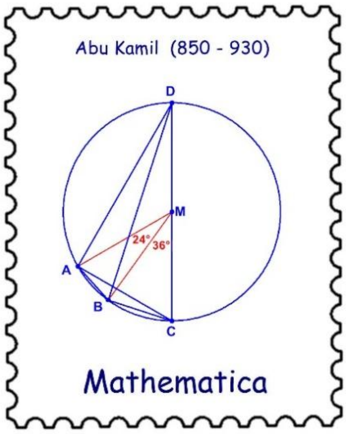
**ABŪ KĀMIL** (ca. 850 – 930)

by HEINZ KLAUS STRICK, Germany

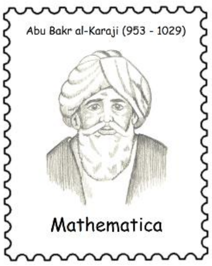



Little information has survived about the mathematician ABŪ KĀMIL SHUJĀ 'IBN ASLAM IBN MUHAMMAD IBN SHUJA : we only know his approximate dates of birth and death, but we do not know how his life unfolded.

In some sources, ABŪ KĀMIL is referred to as AL-HĀSIB AL-MISRĪ , “the calculator from Egypt”; therefore, it is reasonable to assume this to be the country of his origin.

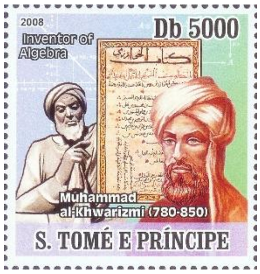
Around the year 987, the copyist and bookseller IBN AN-NADIM published the encyclopedic catalogue *Kitāb al-Fihrist.* His intention was to list the titles of all books published in Arabic up to that point. The list contains the titles of nine books by ABŪ KĀMIL – not all of his works have survived, and some of the texts are only known from Latin and Hebrew translations from later centuries.

ABŪ KĀMIL‘S books had a great influence on subsequent mathematicians, especially ABU BAKR AL-KARAJI (953-1029) and LEONARDO OF PISA, also known as FIBONACCI (1170-1250).










The best known is ABŪ KĀMIL's book on algebra (*Kitāb fī'l-jabr wa'l muqābala*) , also written as a commentary on MUHAMMAD AL-KHWĀRIZMĪ 's work *Al Kit ā b al-Muhtasar fī his ā b al-gabr w-al-muq ā bala*. In it he considers the same six basic types of linear and quadratic equations as his predecessor, adopts his terms and uses the same examples. Like AL-KHWĀRIZMĪ, he explains the transformations of equations using geometric figures by applying theorems from EUCLID's *Elements*. What we write down today in short formal notation is described throughout in words by ABŪ KĀMIL.

More systematically than AL-KHWĀRIZMĪ, ABŪ KĀMIL deals with rules for the signs of multiplication (written in modern notation):

$(+a)\cdot(+b)=+(a\cdot b)$ , $(-a)\cdot(+b)=-(a\cdot b)$ and $(-a)\cdot(-b)=+(a\cdot b)$,

which are required for algebraic transformations, such as

$(a\pm px)\cdot(b\pm qx)=ab\pm bpx\pm aqx+pqx^2$ and

$(a\pm px)\cdot(b\mp qx)=ab\pm bpx\mp aqx-pqx^2$ ,

and also rules like

$\frac{a}{b}\cdot\frac{b}{a}=1$, $\frac{a}{b}+\frac{b}{a}=\frac{a^2+b^2}{a\cdot b}$ , $\frac{a+b}{a}\cdot\frac{a+b}{b}=\frac{a+b}{a}+\frac{a+b}{b}$ and $\sqrt{a\cdot b}=\sqrt{a}\cdot\sqrt{b}$ , $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$ .

Abū Kāmil proves that he has no difficulty calculating with roots. Using the geometrically based relationship, $\sqrt{a} \pm \sqrt{b} = \sqrt{a+b \pm 2\cdot\sqrt{a\cdot b}}$ arithmetic problems such as $\sqrt{18}+\sqrt{8} = \sqrt{18+8+2\cdot\sqrt{144}} = \sqrt{50}$ can be solved, which we usually solve today by extracting partial roots: $\sqrt{18}+\sqrt{8} = 3\sqrt{2}+2\sqrt{2} = 5\sqrt{2} = \sqrt{50}$.

The formula is also of practical use if, for example, you want to know how much $\sqrt{5}\cdot\sqrt{13}$ or $\sqrt{5}+\sqrt{13}$ is *approximately*: $\sqrt{5}\cdot\sqrt{13} = \sqrt{65} \approx 8$, $\sqrt{5}+\sqrt{13} \approx 6$.

For the three types of mixed quadratic equations, Abū Kāmil not only gives the solution terms, but also the square of the solution: For example, in the case of the equation type $x^2 + px = q$ he gives the term $x^2 = \frac{p^2}{2} + q - \sqrt{p^2q + (\frac{p^2}{2})^2}$.

Among the tasks in the algebra book, one repeatedly finds the question of decomposing the number 10 into two summands with certain properties. While in Al-Khwārizmī there is always a natural number on the right, Abū Kāmil does not shy away from putting an irrational number there, as the following example shows:

- The number 10 is to be divided into two summands *x* and $10-x$ such that $\frac{x}{10-x} + \frac{10-x}{x} = \sqrt{5}$.

Multiplication with the main denominator and further transformations lead to the quadratic equation $x^2 + \sqrt{50000} - 200 = 10x$ with the solutions $x = 5 \pm \sqrt{225 - \sqrt{50000}}$.

Abū Kāmil then presents another solution: The substitution $y = \frac{10-x}{x}$ leads to the quadratic equation $y^2 + 1 = \sqrt{5}y$ and thus to the intermediate result $y = \sqrt{\frac{5}{4}} - \frac{1}{2}$. To eliminate the irrational coefficient $\sqrt{\frac{5}{4}}$ in the linear equation $10 - x = (\sqrt{\frac{5}{4}} - \frac{1}{2})\cdot x$, both sides of the equation are squared, which results in the quadratic equation $x^2 + 10x = 100$, from which the solution then $x = \sqrt{125} - 5$ arises.

Abū Kāmil's transformations in the equation $(\sqrt{\frac{x}{2}} + 3)\cdot(\sqrt{\frac{x}{3}} + 2) = 20$ for which he determines the solution $x = 15 + \sqrt{2400} - \sqrt{1449 + \sqrt{2160000}}$ are also tricky.

Al-Khwārizmī's book contains forty tasks; Abū Kāmil's work contains a total of 69 tasks – one of the last tasks is as follows:

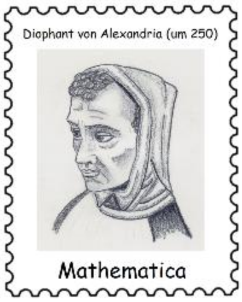
- The number 10 is to be broken down into three terms such that when the small part multiplied by itself is added to the middle part which is also multiplied by itself, the result is the large part multiplied by itself. When the small part is multiplied by the large part, it is equal to the middle part multiplied by itself .

This is about solving the system of equations

$x + y + z = 10;\ x^2 + y^2 = z^2;\ x\cdot z = y^2$ with $x < y < z$.

Abū Kāmil uses the method of false position (*regula falsi*) to solve the problem by first setting $x = 1$, then deriving values for *y* and *z* and then correcting them *proportionally*.
Finally, $x = 5 - \sqrt{\sqrt{3125} - 50}$ and $z = 2\frac{1}{2} + \sqrt{31\frac{1}{4}} - \sqrt{\sqrt{781\frac{1}{4}} - 12\frac{1}{2}}$ and from this one gets an expression for *y*.

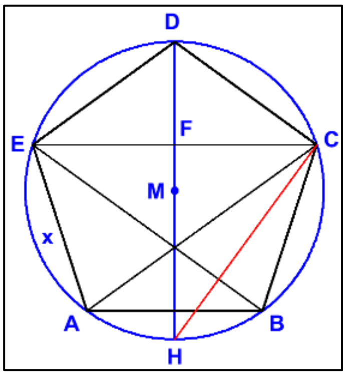
In another work, the *Book of Rarities in Arithmetic* (*Kitāb taraīf fī'l-hisāb*), Abū Kāmil deals with under-determined linear systems of equations with integer solutions. It is noteworthy that Abū Kāmil wrote the book before an Arabic translation of Diophantus's *Arithmetica* had been made; the methods he used differ from those of Diophantus.




The six tasks in the book are formulated as tasks about birds. Abū Kāmil shows the variety of possibilities in terms of the number of solutions:

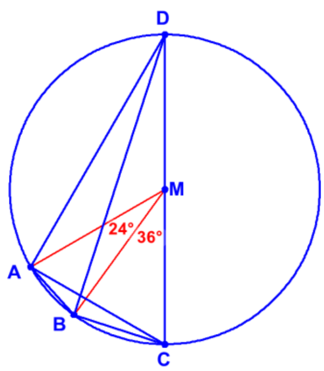
- Buy 100 birds for a total of 100 *dirhams*: one duck costs 5 *dirhams*, one chicken 1 *dirham*, 20 sparrows together cost 1 *dirham*.
  Associated system of equations: $x+y+z=100$; $5x+\frac{1}{20}y+z=100$; Solution: $x=19$ ducks, $z=1$ chicken, $y=80$ sparrows.
- Buy 100 birds for a total of 100 *dirhams*: one duck costs 2 *dirhams*, two pigeons or three wood pigeons or four larks or a chicken each cost 1 *dirham*.
  *Solution* : The system of equations with 5 variables has a total of 2678 solutions, which can be determined using tables of values.
- The system of equations $x+y+z=100$; $\frac{1}{3}x+\frac{1}{2}y+2z=100$ has the following six solutions: $(51;10;39)$, $(42;20;38)$, $(33;30;37)$, $(24;40;36)$, $(15;50;35)$, $(6;60;34)$.
- The system of equations $x+y+z=100$; $3x+\frac{1}{20}y+\frac{1}{3}z=100$ has *no* solution.

Abū Kāmil's *Book on the Pentagon and the Decagon* contains twenty problems. The first four are about determining a regular pentagon or decagon given the diameter of the inner or circumcircle.




*Example* : Determining the side length of an inscribed regular pentagon with circumradius $R=5$.
From the figure on the right we get: $DC^2=x^2=DF\cdot DH=10\cdot DF$ (according to Pythagoras's theorem for the triangle *HCD*), so $DF=\frac{x^2}{10}$, and $FC^2=CD^2-DF^2=x^2-\frac{x^4}{100}$ (according to Pythagoras's theorem in the triangle *FCD*), so $EC^2=4x^2-\frac{1}{25}x^4$.
Furthermore, in the symmetrical trapezoid *ABCE* (*according to* Ptolemy 's theorem: $AC\cdot BE=AB\cdot CE+BC\cdot EA=x\cdot CE+x^2$.

Because $EC=EB=AC$ it follows that $EC^2=4x^2-\frac{1}{25}x^4=x\cdot CE+x^2$ and therefore $EC=3x-\frac{1}{25}x^3$, so $EC^2=(3x-\frac{1}{25}x^3)^2$.
Together with $EC^2=4x^2-\frac{1}{25}x^4$ and after transformations we get the biquadratic equation $x^4=125x^2+3125$ with the solution $x=5\cdot\sqrt{\frac{5-\sqrt{5}}{2}}=\sqrt{62,5-\sqrt{781,25}}$.




Further tasks deal with the relationship between the radii of the inner and outer circumference and the side length of regular polygons as well as with the determination of a regular 15-gon, see figure on the right:

If *A*, *C* are vertices of a regular hexagon and *B*, *C* are vertices of a regular decagon, then *A*, *B* are vertices of a regular pentadecagon. Using Ptolemy's theorem applied to the quadrilateral *ABCD,* we get $R=5$:

$$AB=\sqrt{15,625+\sqrt{48,828125}}+\sqrt{4,6875-\sqrt{23,4375}}$$

When faced with the task of determining the side length of an equilateral triangle where the sum of the area and height is equal to 10, ABU KAMIL abandons the previous tradition of only considering sums of equidimensional quantities – this is obviously not about an application, but only a disguised arithmetic task.

ABU KAMIL'S surviving works include the book *On the Art of Measurement*. Since this work was written for everyday use, especially for craftsmen and surveyors, it contains no derivations or proofs, but only formulae and numerical examples for calculating sides, areas and circumferences of simple geometric figures (triangles, quadrilaterals, regular polygons, circles and parts of circles) as well as surfaces and volumes of various bodies.

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