PIETRO MENGOLI (1626 – June 7, 1686)

by HEINZ KLAUS STRICK, Germany

Little is known about PIETRO MENGOLI's origins, nor is his exact date of birth likely to be found out. He spent his entire life in Bologna, which at that time belonged to the Papal States.

At the University of Bologna he attended lectures by BONAVENTURA CAVALIERI. When the latter died in November 1647, MENGOLI was commissioned to take over his lectures in arithmetic.

In 1650 MENGOLI received his doctorate in philosophy and three years later also in civil and canon law and in parallel he gave lectures on mechanics.

From 1668 until his death he held the chair of mathematics.

Ordained as a priest, he took over the parish of Santa Maria Magdalena and the management of an associated monastery from 1660 onwards, which took up so much of his time that he did not publish anything again until 1670.

The publications from the 1650s and 1670s dealt, among other things, with infinite series and with area determinations (see below), with EUCLID's doctrine of proportions and with the refraction and parallax of solar rays.

In one of the writings he dealt with GALILEO 's theory on the question of how music is heard (with the speculative assumption of a second eardrum in the ear).

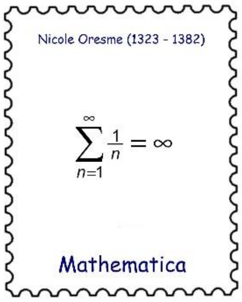
In his last works (*Arithmetica rationalis* and *Arithmetica realis*) he attempted to build up a logical, physical and metaphysical system on a mathematical basis, through which a rational justification of the Catholic doctrine of faith should become possible. In doing so, he was in close correspondence with the influential Cardinal LEOPOLDO DE MEDICI since the trial of GALILEO GALILEI and his conviction was still a present threat decades later, especially in the minds of the scientists living in the Papal States.

Since MENGOLI wrote his writings in a strange Latin that was difficult to understand, interest in his publications quickly waned. Nevertheless, there was positive feedback, among others from the Secretary of the *Royal Society*, HENRY OLDENBURG, who was particularly interested in MENGOLI's music theory. It was only in the 20th century that it became clear that the Italian mathematician was far ahead of his time in some of his considerations.

One of these topics was the study of infinite series. Since antiquity, arithmetic with geometric sequences and the corresponding partial sum sequences, the geometric series, had been known:

For $a \in IR$ and $0 < q < 1$, the sequence of numbers $a, aq, aq^2, aq^3, ...$ converges to zero and the sequence of partial sums $(a + aq + aq^2 + aq^3 + ... + aq^n)_{n \in IN}$ converges to the number $\frac{a}{1-q}$.

MENGOLI now stated that from "*The sequence* $(a_n)_{n \in IN}$ *converges to zero*" it does not necessarily follow that "*The associated sequence* $(\sum_{k=1}^{n} a_n)_{n \in IN}$ *of the partial sums converges to a* (*finite*) *number*".



Although the French mathematician and philosopher NICOLE ORESME had already proved 300 years earlier that this was not true for the so-called *harmonic series*, i.e. the sequence of the partial sums of the reciprocals of the natural numbers $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n}$, his proof was forgotten and was also unknown to MENGOLI.

ORESME had shown with the help of a diverging sequence smaller than H_n, that H_n grows beyond all limits:

$H_1 = 1;\ H_2 = 1 + \frac{1}{2} = 1.5;\ H_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2;$

$H_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 2.5;$

$H_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \ldots + \left(\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \ldots + \frac{1}{8}\right) + \left(\frac{1}{16} + \ldots + \frac{1}{16}\right) = 3$ etc.

MENGOLI, in his book *Novae quadraturae arithmeticae, seu de additione fractionum*, published in 1650, gave an indirect proof of this property, i.e. he made the approach:

Suppose the series has a finite limit value $H = 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) + \ldots$.

Now, for three consecutive fractional terms $\frac{1}{a-1} + \frac{1}{a} + \frac{1}{a+1}$, it is true that they are greater than three times the fraction in the middle:

$\frac{1}{a-1} + \frac{1}{a} + \frac{1}{a+1} = \frac{1}{a} + \frac{a+1+a-1}{(a-1)(a+1)} = \frac{1}{a} + \frac{2a}{a^2-1} > \frac{1}{a} + \frac{2a}{a^2} = \frac{1}{a} + \frac{2}{a} = \frac{3}{a}$, for example,

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{1}{3} + \frac{3}{4} > \frac{1}{3} + \frac{6}{9} = \frac{1}{3} + \frac{2}{3} = 1$ and $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{1}{6} + \frac{12}{35} > \frac{1}{6} + \frac{12}{36} = \frac{1}{6} + \frac{2}{6} = \frac{1}{2}$.

Therefore, one can estimate as follows:

$H > 1 + \frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \ldots = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \ldots = 1 + H$

Since the positive finite number H cannot be greater than $1 + H$, the assumption that H is a finite number must be false. Thus it is proved that the harmonic series has no finite limit, i.e. is divergent.

MENGOLI also investigated the *alternating harmonic series* in his book and found that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln(2)$. This result was also proved 18 years later and independently of MENGOLI by NICOLAUS MERCATOR.

Further, in the *Novae quadraturae* one finds the proof that the sum of the reciprocals of the sequence of triangular numbers converges to the limit 2.

He first showed that $\frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \ldots + \frac{1}{\frac{1}{2} \cdot n \cdot (n+1)} = \frac{n-1}{n+1}$ holds and then that the difference $1 - \frac{n-1}{n+1}$ can be smaller than any positive number, no matter how small, if only *n* is chosen large enough – a description that comes very close to today's notion of limit.

In a next step, MENGOLI dealt more generally with partial sum sequences whose summands are reciprocals of products of natural numbers: $\frac{1}{1 \cdot (1+r)} + \frac{1}{2 \cdot (2+r)} + \frac{1}{3 \cdot (3+r)} + \ldots + \frac{1}{n \cdot (n+r)}$.

For *r* = 1 you get half of the last sequence considered.

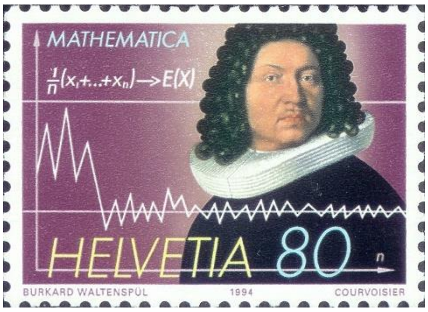
For *r* = 2 we have $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \ldots + \frac{1}{n \cdot (n+2)}$ with limit $\frac{3}{4}$, for $r = 3$ we have $\frac{1}{4}+\frac{1}{10}+\frac{1}{18}+...+\frac{1}{n\cdot(n+3)}$ with limit $\frac{11}{18}$,

for $r = 4$ we have $\frac{1}{5}+\frac{1}{12}+\frac{1}{21}+...+\frac{1}{n\cdot(n+4)}$ with limit $\frac{25}{48}$ and so on.

All partial sum sequences of this type are convergent; MENGOLI was able to prove that the following applies to the limit:

$$\frac{1}{1\cdot(1+r)}+\frac{1}{2\cdot(2+r)}+\frac{1}{3\cdot(3+r)}+...=\frac{1}{r}\cdot(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{r}) .$$

He tried in vain to solve the special case $r = 0$. The proof that the sequence of the partial sums of the reciprocal square numbers $Q_n = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+...+\frac{1}{n^2}$ converges and that the limit $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+...=\frac{\pi^2}{6}$ holds was only achieved 85 years later by LEONHARD EULER.



Previously, mathematicians at the University of Basel, including JACOB BERNOULLI, had tried in vain to determine the limit (the so-called *Basel problem*).

Incidentally, for a long time it was erroneously assumed that it was JACOB BERNOULLI who first proved the divergence of the harmonic series, until it was discovered that ORESME and MENGOLI had beaten him to it.

MENGOLI also investigated reciprocals of products of three consecutive natural numbers; among other things, he showed that

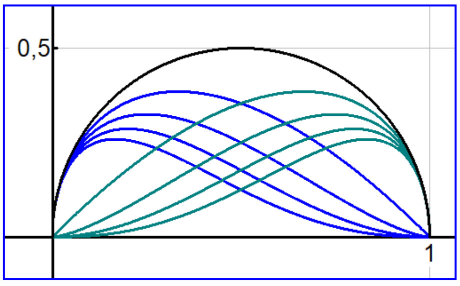
$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+...=\frac{1}{6}+\frac{1}{24}+\frac{1}{60}+...=\frac{1}{4}.$$

In his 1659 work *Geometriae speciosae elementa*, he continued his investigations into convergent and divergent sequences; in principle, he discovered the limit theorems for sums, products and quotients.

He decomposed surfaces by means of inscribed and circumscribed parallelograms and proved that the corresponding partial sum sequences had common limits; the influence of his approach on WALLIS and LEIBNIZ is unmistakable.

In a paper published in 1672 work *Circulo*, he explored which areas are enclosed under graphs of the type $\sqrt{x^m\cdot(1-x)^n}$.

(In the figure on the right, the graphs of $y=\sqrt{x^1\cdot(1-x)^n}$ and $y=\sqrt{x^m\cdot(1-x)^1}$ with $1\le m,n\le 5$ are shown.)



Inspired by publications of the French mathematician JACQUES OZANAM, MENGOLI dealt with special DIOPHANTINE equations, i.e. equations with integer solutions, in the 1670s.

OZANAM had posed the following 6-square problem, among others:

- *We are looking for three natural numbers x, y, z whose differences* $x-y$, $x-z$, $y-z$ *are square numbers and the differences* x^2-y^2, x^2-z^2, y^2-z^2 *of the squares are also square numbers.*

(*Note*: OZANAM later also posed an analogous problem with sums instead of differences.)

MENGOLI tried to prove that there were no solutions to the problem; but when OZANAM presented a solution triple (2 288 168, 1 873 432, 2 399 057), he saw his reputation endangered and set out again on a search. And after intensively studying the properties of PYTHAGOREAN number triples, he finally found two more solutions.

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