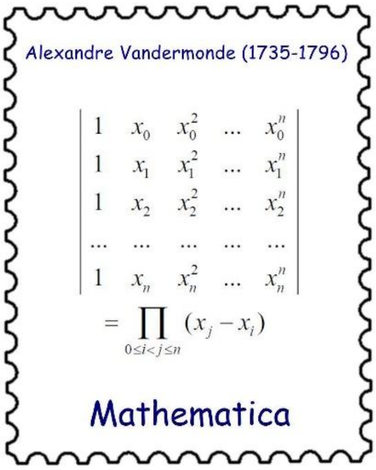
ALEXANDRE-THÉOPHILE VANDERMONDE (February 28, 1735 – January 1, 1796)

by HEINZ KLAUS STRICK , Germany



He had diverse interests and talents; he was one of the active and influential figures during the years of the French Revolution. He was a lawyer, musician, mathematician, chemist, physicist, mechanic, and, at the end of his life, the holder of the first official chair of political economy in France. Yet no portrait of him exists.

A special form of determinant (see the Mathematica stamp on the right) and an equation from combinatorics are named after him – although he did not specify either in the way that is common today.

We are talking about ALEXANDRE-THÉOPHILE VANDERMONDE, born in Paris in 1735, where he also died on New Year's Day 1796.

His father was the physician JACQUES-FRANÇOIS VANDERMONDE, who served the French *Compagnie des Indes* in Macau until 1732. His father had a son named CHARLES in 1727 from his first marriage and ALEXANDRE-THÉOPHILE was from the physician's second marriage, which he entered into after returning to France. CHARLES, like his father, studied medicine. He died on the eve of his wedding in 1762. ALEXANDRE-THÉOPHILE remained unmarried and childless.

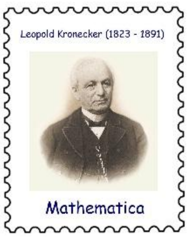
The somewhat sickly ALEXANDRE-THÉOPHILE studied law at the University of Paris and passed both state exams. Thanks to the fortune of his father, who had died in 1746, and later as an heir of his half-brother, he was not dependent on practising a profession. Initially, he devoted himself primarily to music and also to a possible career as a very gifted violinist.

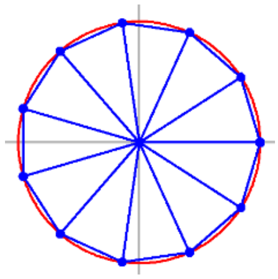
He maintained close contact with the encyclopedists DENIS DIDEROT and JEAN-BAPTISTE LE ROND D'ALEMBERT, and thus also became acquainted with contemporary mathematicians.

Somewhat surprisingly, he was admitted as a member of the *Académie des Sciences* in 1771. This was due to his treatise, *Mémoire sur la résolution des équations*, in which he looked for ways to find formulas for solving equations of the fifth degree and higher.

However, his approach only allowed him to solve specific equations.

The significance of VANDERMONDE's approach was larely overlooked in the following years due to the rapid development of the theory.





It was only in 1888 that Leopold Kronecker recognised in retrospect that Vandermonde had laid the foundation for modern algebra. The advances made in the solvability of equations, subsequently achieved through Joseph-Louis Lagrange's resolvent method and later by Niels Hendrik Abel and Evariste Galois, had its beginnings in Vandermonde's contribution. In the course of his investigations, he succeeded in being able to solve certain cyclotomic equations, including $x^{11} - 1 = 0$. The connection with the constructibility of regular polygons was first recognised by Carl Friedrich Gauss.

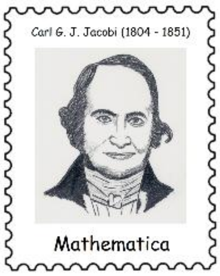
In the following years, Vandermonde submitted another treatise to the *Académie* (*Mémoire sur l'élimination*), in which he searched for a general solution formula for linear systems of *n* equations. As early as the end of the 17th century, the Japanese mathematician Seki Kowa Takakazu (1683) and Gottfried Wilhelm Leibniz (1693) - working independently - had already investigated matrices as coefficient schemes for systems of linear equations and used the associated determinants to solve them.

Vandermonde was the first to consider determinants as alternating, multilinear functions and systematically investigated their properties. Among other things he found:

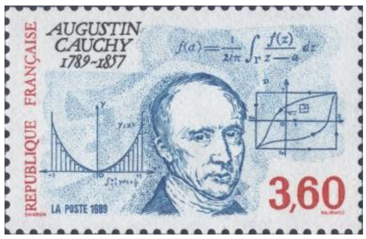
Properties of determinants

- If you swap two rows or two columns of a matrix, the sign of the determinant changes; that is, if there is an even number of swaps, the sign does not change.
- If two rows or two columns of a matrix are identical, then the determinant is zero.
- The value of a determinant does not change when you add a multiple of one row (column) to another row (column).

However, what is understood today as a Vandermonde determinant is not found in the above mentioned *Mémoire* or in any of his other papers.



The historically incorrect attribution may have arisen because both Augustin Cauchy and Carl Gustav Jakob Jacobi repeatedly cited to Vandermonde's contributions in connection with the theory of matrices and determinants, and this was then repeatedly adopted by others with incorrect references until the term was finally established in the 20th century.



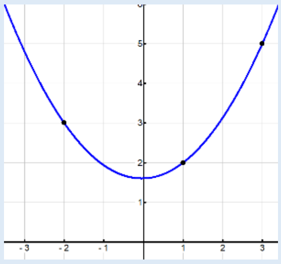
Application of the Vandermonde determinant in the case $n=3$

$$\begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = x_1 \cdot x_2^2 + x_0 \cdot x_1^2 + x_2 \cdot x_0^2 - x_1 \cdot x_0^2 - x_2 \cdot x_1^2 - x_0 \cdot x_2^2 = (x_1 - x_0)\cdot(x_2 - x_0)\cdot(x_2 - x_1)$$

Problem: Find the coefficients of a quadratic function $p(x) = a + bx + cx^2$ whose graph passes through the following points: (-2, 3), (1, 2) and (3, 5).

If the values are entered $x_0 = -2, x_1 = 1, x_2 = 3$ into the VANDERMONDE matrix and the corresponding determinant is calculated, it is found to be non-zero (see right). This means that the linear system of equations has a unique solution.

$$\begin{vmatrix} 1 & -2 & 4 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{vmatrix} = 6 \neq 0$$



The coefficients result from the linear system of equations:

$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{pmatrix} \circ \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \Leftrightarrow \left\{ \begin{matrix} a - 2b + 4c = 3 \\ a + b + c = 2 \\ a + 3b + 9c = 5 \end{matrix} \right\} \Leftrightarrow \ldots \Leftrightarrow \left\{ \begin{matrix} a = \frac{8}{5} \\ b = \frac{1}{30} \\ c = \frac{11}{30} \end{matrix} \right\}$$

A third work by VANDERMONDE, entitled *Remarques sur les problèmes de situation* (Remarks on the problems of position), dealt with the knight's move problem on the chessboard. It also contained ideas that today would be considered as graph theory.

In his fourth work, *Mémoire sur des irrationnelles des différents ordres avec une application au cercle* (Notes on irrational numbers of different orders with an application to the circle), he investigated products with *k* decreasing factors: $[n]^k = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+1)$ or equivalently $[n]^k = \frac{n!}{(n-k)!}$. In particular $[n]^0 = 1, [n]^1 = n, \ldots, [n]^n = n!$ (here denoted by the factorial symbol '!', as is common today, though this was only "invented" in 1808 by the Alsatian mathematician CHRISTIAN KAMP).

VANDERMONDE further defined $[n]^{-k} = \frac{1}{(n+1)\cdot(n+2)\cdot(n+3)\cdot\ldots\cdot(n+k)} = \frac{1}{[n+k]^k}$ $\left(= \frac{n!}{(n+k)!}\right)$,
where $[0]^{-k} = \frac{1}{k!}$, so $[0]^{-1} = 1, [0]^{-2} = \frac{1}{2}, [0]^{-3} = \frac{1}{6}, \ldots$.

He then extended these definitions to fractions *n*, *k,* resulting in the following representation $\frac{\pi}{2} = [\frac{1}{2}]^{\frac{1}{2}}\cdot[-\frac{1}{2}]^{-\frac{1}{2}}$ for WALLIS 's product $\frac{\pi}{2} = \frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot\ldots}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot\ldots}$.

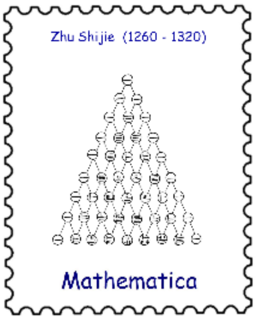
He further showed: $\sqrt{\pi} = 2\cdot[\frac{1}{2}]^{\frac{1}{2}}$, $[\frac{1}{2}]^{\frac{1}{3}}\cdot[-\frac{1}{2}]^{-\frac{1}{3}} = \frac{3}{2}$ and $[\frac{1}{2}]^{\frac{1}{4}}\cdot[-\frac{1}{2}]^{-\frac{1}{4}} = \sqrt{2}$.

Binomial coefficients can also be written using this method:

$$\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!} = [n]^k\cdot[0]^{-k}.$$

The following holds for the binomial series:

$$(1+x)^n = 1+[n]^1 \cdot [0]^{-1} \cdot x + [n]^2 \cdot [0]^{-2} \cdot x^2 + [n]^3 \cdot [0]^{-3} \cdot x^3 + \ldots$$



It is no longer possible to trace by whom and from when the equation $\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$ was given the name *Vandermonde identity* .

In English-speaking countries, it is referred to as the *Zhu-Vandermonde identity,* as the relationship was already known to the Chinese mathematician Zhu Shijie (1260-1320). However, it is not found in Vandermonde's writings or surviving manuscripts.

This identity can be proved in different ways:

Algebraic :

$$(1+x)^r \cdot (1+x)^s = \left[\sum_{i=0}^{r} \binom{r}{i} \cdot x^i\right] \cdot \left[\sum_{j=0}^{s} \binom{s}{j} \cdot x^j\right] = \sum_{k=0}^{r+s} \binom{r+s}{k} \cdot x^k = (1+x)^{r+s}$$

Combinatorially :

k balls are drawn from an urn containing *r* red and *s* black balls. There are this many possible compositions for this sample:

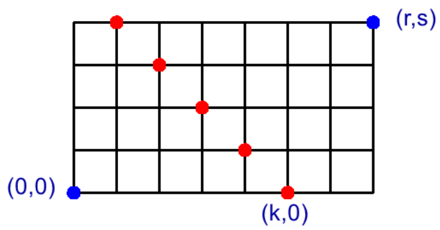
$$\binom{r}{0} \cdot \binom{s}{k} + \binom{r}{1} \cdot \binom{s}{k-1} + \binom{r}{2} \cdot \binom{s}{k-2} + \ldots + \binom{r}{k} \cdot \binom{s}{0} = \binom{r+s}{k}$$

Geometric :

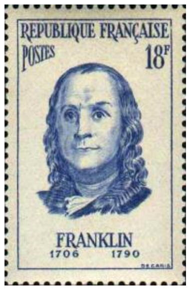
The number of paths of length $r+s$ on a grid from the origin to the point (r,s).

After *k* steps an interim tally is taken:

The number of paths to the intermediate point is multiplied by the number of remaining paths.



Vandermonde devoted himself to other areas. In 1778 he presented the *Académie* with a new system of harmony, which he had previously discussed with various composers, including Christoph Willibald Gluck and François-André Danican Philidor (the well-traveled Philidor was also considered the world's best chess player at that time).



Professional opinions about Vandermonde were divided. The musicians considered him an outstanding mathematician, the mathematicians appreciated him as an outstanding musician.

Vandermonde was convinced that the *Glass Harmonica* developed by Benjamin Franklin (inventor of, among other things, the lightning rod) would be particularly suitable for realising his musical ideas (the *glass harmonica* consists of rotating, chromatically tuned glasses where tones are produced by rubbing with the finger).

In 1776, he conducted experiments at low temperatures with Étienne Bézout and Antoine Lavoisier. Together with Gaspard Monge and Claude-Louis Berthollet, he tested various combinations of iron and carbon to improve the quality of steel. His collaboration with Monge was so close, in fact, that some cynics called Vandermonde "*femme de Monge*" (Monge's wife).

He was also interested in the automata invented and collected by the engineer Jacques de Vaucanson. When Vaucanson died in 1783, Vandermonde took over the collection (*Cabinet des Mécaniques du Roi*), which from 1794 belonged to the *Conservatoire national des arts et métiers*, in whose management he also participated. (In 1804, Joseph-Marie Jacquard, inventor of the punch-card-controlled automatic loom, took over the directorship of the museum.)

Convinced by the ideas during the French Revolution, Vandermonde was involved in the founding of important institutions of the young republic. He took over the office for the clothing of the armed forces, was a member of the *Commission for Weights and Measures* (introduction of the metric system) and the Commission for the Protection of Monuments (*Commission des monuments*).

Together with Pierre-Simon Laplace, Joseph-Louis Lagrange and Gaspard Monge, he took care of the content and structural design of the newly founded *École Normale Supérieure* in 1794, where he took over a professorship in political economy in February 1795.

However, due to his tuberculosis, he was only able to give eight lectures and then he was exhausted.

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https://mathematik-ist-schoen.jimdoweb.com/

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