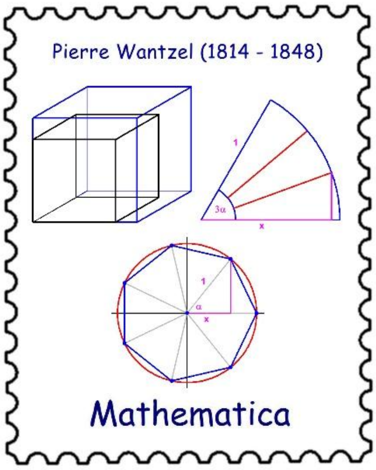
PIERRE WANTZEL (June 5, 1814 – May 21, 1848)

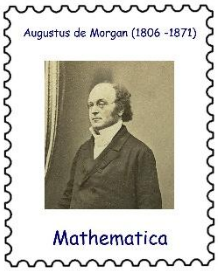
by HEINZ KLAUS STRICK, Germany

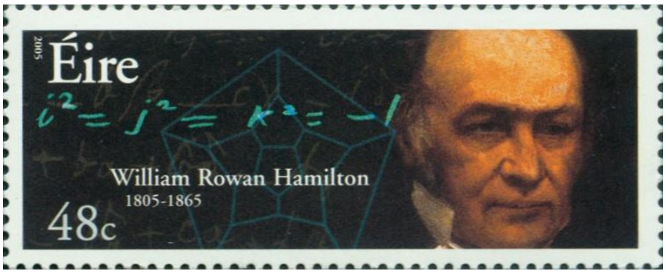


For over two millennia, mathematicians tried in vain to solve four "classical" geometric problems using a construction with compass and (unmarked) ruler:

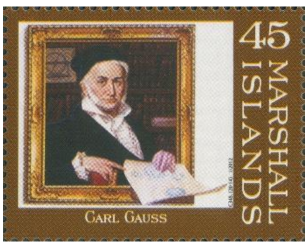
- to construct the side length of a cube whose volume is twice that of the unit cube,
- to divide any angle into three equal partial angles by construction,
- to construct a regular polygon with any number of vertices,
- to construct a square that has the same area as the unit circle.

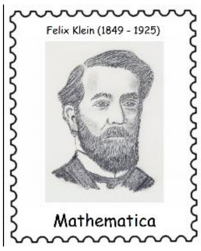
When in 1837 a 23-year-old Frenchman solved the first three of these four problems by proving the *fundamental* impossibility of a construction, this was done without much fanfare in the scientific community, even though his seven-page paper appeared in one of the most prestigious journals of the time, the *Journal de mathématiques pures et appliquées* (also called *Liouville's Journal*).





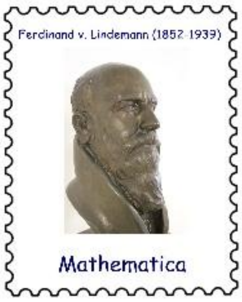
Even 15 years after the publication of this article, WILLIAM ROWAN HAMILTON was still corresponding with AUGUSTUS DE MORGAN about the question of whether a construction might one day be found to divide any angle into thirds, since "... 100 years ago no one would have believed that it was possible to construct a regular 17-gon...", which CARL FRIEDRICH GAUSS had succeeded in doing in 1796.





Even the geometry expert FELIX KLEIN was unsure for decades afterward whether rigorous proofs had actually been published in the meantime. In the *Encyclopedia of Elementary Mathematics* by H. WEBER and J. WELLSTEIN from 1905, it is stated in general terms: "*Only since the foundation of modern algebra by Gauss and Abel has it been possible to rigorously prove that the trisection of angles and the construction of regular polygons can only be carried out exactly in certain exceptional cases using compass and straightedge.* " However, it is not stated that this has been proven. It was not until 1937, exactly 100 years after his publication, that PIERRE WANTZEL is described in JOHANNES TROPFKE's *History of Elementary Mathematics* as the first to present exact proofs.

Why did it take a century for Wantzel's publication to be noticed? Firstly, the author was virtually unknown – and remained so, as he only lived to the age of 33 and did not have many opportunities to become known until his early death. Secondly, these impossibility proofs were probably not considered important, since Gauss had shown in his *Disquisitiones Arithmeticae* of 1801 with regard to the constructibility of regular *n*-gons that they are possible using only compass and straightedge, if *n* can be represented as the product $n = 2^r \cdot p_1 \cdot p_2 \cdot ... \cdot p_s$ of a power of 2 with *mutually different* Fermat primes (these are prime numbers of the form $p_k = 2^{2^k} + 1$, i.e., 3, 5, 17, 257, ...). Gauss then further claimed that no other regular *n*-gons could be constructed, but, as he wrote, he omitted a proof due to lack of space. With his suggestion, he wanted to save others the trouble of searching for a construction that, in his opinion, did not exist. In this respect, Wantzel's article seemed to contain little new, even though Gauss hadn't actually provided the proof of non-constructibility for the other numbers of vertices.



Ferdinand von Lindemann was later able to prove in 1882 that squaring the circle was not possible constructively, the topic of the "classical problems of antiquity" was finally considered "closed" in mathematical research.

Despite the publication of Tropfke (see above), the merits of Pierre Wantzel were often not mentioned in later mathematics books, but rather the authors tended to give the credit to Gauss, Abel or even Lindemann.

Pierre Laurent Wantzel's family, on his father's side, originally came from Germany. His father, Frédéric, joined the French army in 1814 – three months before Pierre's birth – and served for seven years. During this time, Pierre lived with his mother and her parents in Ecouen (near Paris). When his father returned, he took a position as a mathematics professor at the *École Spéciale du Commerce*. During his elementary school years, Pierre lived with his teacher, who was also a surveyor. Even then, his exceptional comprehension was evident, and he was even able to help solve difficult surveying problems.

Pierre transferred to the *École des Arts et Métiers* in Châlons; however, he felt uncomfortable there and under-challenged, as the emphasis was more on manual skills. After earnest pleas, his father allowed him to transfer to a school in Paris, where the director, M. Lievyns, taught him the Latin and Greek he lacked before he was finally able to transfer to the *Collège Charlemagne* in 1828. (M. Lievyns, incidentally, became his father-in-law 14 years later.)

At his new school, his diverse talents became apparent: first prizes in the school's Latin and French competitions, second prize in the Paris Latin competition, and then first prizes in the nationwide physics and mathematics competition (*Concours Général*). He assisted his mathematics teacher by correcting the proofs for his "*Traité d'Arithmétique*."

At the age of 18, Pierre passed the entrance exams to both the *École Polytechnique* and the *École Normale Supérieure* as top of his class, something no one had ever achieved before. After successfully completing his studies at the *École Polytechnique,* Wantzel transferred to the *École Nationale des Ponts et Chaussées* in 1834 (which was quite common).

However, as he did not want to become a "mediocre engineer" and was instead looking for the opportunity to concentrate more intensively on mathematics, he wanted to take a leave of absence – with the result that he was asked to work as a tutor at the university. He took on this role at the *École Polytechnique* in parallel, and later the same university also asked him to administer the entrance exams. He repeatedly took on a substitute teaching role at the *Collège Charlemagne* – with great success. After being awarded the title of engineer in 1840, he travelled throughout the country as an inspector of technical equipment, allowing himself no breaks.

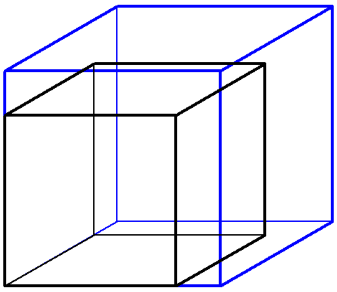
Despite the long, strenuous working days, Wantzel wrote a number of scientific articles, including on the curvature behavior of elastic rods and the laws of fluid pressure, such as those prevailing at canal locks, on the integration of certain integral equations, and, last but not least, his contribution to constructibility with compass and straightedge. In 1845, he published a simplification of Niels Hendrik Abel's proof regarding the impossibility of solving fifth-degree equations in general using radicals.

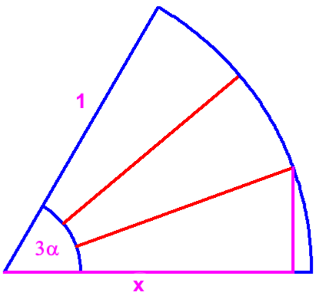
In his restlessness, he failed to notice how he was ruining his health; he overcame his fatigue with coffee and opium and ate only irregular meals. This hardly changed after he married – he died at the age of only 33, without having adequately provided for his widow and two daughters.

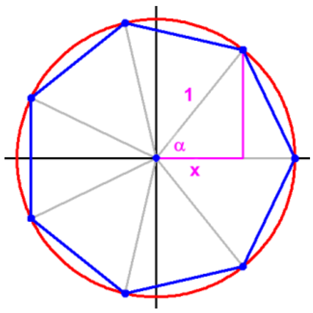
In his 1837 essay *Recherche sur les moyens de reconnaître si un problème de géométrie peut se résoudre à la règle et au compas* (*Investigation into ways of determining whether a geometric problem can be solved with ruler and compass*) Wantzel had proved the following theorem:

- A real number *r* (a segment of length $|r|$) can be constructed with compass and (unmarked) ruler *if and only if it satisfies an irreducible equation of degree* 2^k ($k > 1$).

Starting from a unit line on the number line (x -axis), the elementary constructions, as described in Euclid's *Elements*, lead in the first step to points whose coordinates appear as the solution of a linear or quadratic equation; in a subsequent construction step, only coordinates that are zeros of polynomials of the 1st, 2nd, or 4th degree can appear, etc.







- Doubling the unit cube results in a cube with side length $\sqrt[3]{2}$; the corresponding minimal polynomial is $x^3 - 2$, i.e., a 3rd degree polynomial.
- When trisecting an angle, one considers the addition theorem $\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha)$; e.g., for $3\alpha = 60°$ and $x = \cos(\alpha)$ the equation $\frac{1}{2} = 4x^3 - 3x$ is to be solved and the corresponding minimal polynomial is $8x^3 - 6x - 1$, i.e. a polynomial of degree 3 for which no rational solution exists.

- Of the regular *n*-gons, for example, the 7-gon cannot be constructed, since the central angle here α satisfies, $\cos(3\alpha) = \cos(4\alpha)$ i.e. $4\cos^3(\alpha) - 3\cos(\alpha) = 8\cos^4(\alpha) - 8\cos^2(\alpha) + 1$. Using , $x = \cos(\alpha)$ we get the relationship $4x^3 - 3x = 8x^4 - 8x^2 + 1$, after division by the linear factor $(x-1)$ this leads to $8x^3 + 4x^2 - 4x - 1$, again a 3rd degree polynomial.

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